Divisibility Tests in Different Number Systems and Their Applications

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Abstract

The divisibility test is a fascinating topic in number theory. For example, a number is divisible by 2 if and only if its last digit is even; a number is divisible by 3 if and only if the sum of all its digits is divisible by 3. In this thesis project, under the guidance of Dr. Han, I will investigate and derive the divisibility criteria for various numbers. I will then investigate possible applications for the gained knowledge of divisibility. To add more depth to this project, I will further investigate various divisibility criteria in different number systems and their applications. The research will be conducted through rigorous mathematical analysis in the area of number theory.

Divisibility Tests in Different Number Systems and Their Applications

Contents:

1. Introduction

My interest in this topic began when I was in sixth grade. At the time, my teacher called it the divisibility rule, and we learned about it as one of our math lessons. I learned about the rules for the first ten numbers except seven and eight which I looked up on my own later. The divisibility rule became my favorite topic and became the one math skill I would remember and continue to use to this day. These would include looking at digital clocks trying to see what tests I can perform with the numbers on it, or when I calculate my bowling average for three games by finding the closest number divisible by three in my series. As I went through middle and high school, we barely talked about the divisibility rule in any math class unless it was tied to the actual topic. It seemed like it was useless information that I just happened to know in the back of my head. That all changed when I started going to school here at Millersville University. Since I am in the honors college, one of my requirements is to complete and defend a thesis. As I started thinking about ideas, I remembered going to Dr. Han's virtual problem-solving seminars during my freshman year, and I asked him about how much he knew about the divisibility rule. We each talked about what we knew which sparked my interest even more. When I told Dr. Han about wanting to do something with this piece of knowledge, he suggested I take Number Theory with him over the summer. The class only touched the now called divisibility test for one session, but it helped confirm that this is what I wanted to work on. Dr. Han and I started talking about collaborating during the fall semester of my junior year, and we began working since last spring. I have always known how to use the divisibility test, but I have learned through this experience about why we use it and why it works. Throughout this paper, I will provide proofs behind the divisibility test of several numerical values along with examples on how each test works in the decimal system. Later, we will explore how to use these tests on different numerical bases by using the octal system. According to [4] and [5], We mainly use the decimal system which is base 10 because dec is the prefix for 10. It stems from ancient civilization because people felt 10 is the best base to use due to humans typically having ten fingers. Additionally, we have ten single digits from 0 to 9 until the next digit is added or changed.

2. Basic Divisibility Tests

In this chapter, we will go over some basic and well-known divisibility tests for the decimal system and the proofs that follow. Most of these tests are slightly touched on by [1], [2], [3], but these are the more rigorous proofs behind them. Throughout this paper, we use the notation $a_n a_{n-1} \dots a_1 a_0$ to represent an $(n + 1)$ - digit number, not the product of a_0, a_1, ... a_{n-1}, and a_n.

2.1: Divisibility Test for 2 and 5

This test is one of the examples used in [3] to explain if a number is divisible by 2, but the proof below gives a more detailed explanation about the divisibility test of not only 2, but also 5. It is also important to note that 2 and 5 are factors of 10, which is why these two tests are probably the simplest in the decimal system.

Let $a_n a_{n-1} \dots a_1 a_0$ be an $(n+1)$ - digit number. Then $2|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $2|a_0$. Similarly, $5|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $5|a_0$.

Proof:

Since $10^0 \equiv 1 \, mod \, 2$, $a_0 * 10^0 \equiv a_0 * 1 \equiv a_0 \, mod \, 2$.

Similarly, since $10^0 \equiv 1 \, mod \, 5$, $a_0 * 10^0 \equiv a_0 * 1 \equiv a_0 \, mod \, 5$.

Since $10^1 \equiv 0 \, mod \, 2$, $a_1 * 10^1 \equiv a_1 * 0 \equiv 0 \, mod \, 2$.

Similarly, since $10^1 \equiv 0 \, mod \, 5$, $a_1 * 10^1 \equiv a_1 * 0 \equiv 0 \, mod \, 5$.

For $n > 1$, since $10^n \equiv 0 \, mod \, 2$, $a_n * 10^n \equiv a_n * 0 \equiv 0 \, mod \, 2$.

Similarly for $n > 1$, since $10^n \equiv 0 \, mod \, 5$, $a_n * 10^n \equiv a_n * 0 \equiv 0 \, mod \, 5$.

Then,
$$\begin{aligned} a_n a_{n-1} \dots a_1 a_0 &= a_0 * 10^0 + a_1 * 10^1 + \cdots + a_{n-1} * 10^{n-1} + a_n * 10^n \\ &\equiv a_0 + 0 + 0 + \cdots + 0 + 0 \\ &\equiv a_0 \, mod \, 2. \end{aligned}$$

Similarly,
$$\begin{aligned} a_n a_{n-1} \dots a_1 a_0 &= a_0 * 10^0 + a_1 * 10^1 + \cdots + a_{n-1} * 10^{n-1} + a_n * 10^n \\ &\equiv a_0 + 0 + 0 + \cdots + 0 + 0 \\ &\equiv a_0 \, mod \, 5. \end{aligned}$$

Thus, $2|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $2|a_0$, and $5|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $5|a_0$.

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According to the test above, one can determine whether a number is divisible by 2 or 5 by just looking at the last digit of that number.

For 2, the last digit would need to be 0, 2, 4, 6, or 8.

For 5, the last digit must be 0 or 5.

As an example, if we want to know whether 38 is divisible by 2, we check the last digit which is 8. Since 8 is divisible by 2, we know 38 is divisible by 2.

As another example, if we want to know whether 45 is divisible by 5, we check the last digit which is 5. Since 5 is divisible by 5, 45 is divisible by 5.

2.2: Divisibility Test for 4

Similar to the divisibility test for 2, this test has been slightly touched on from [3], but below is a rigorous proof behind it.

Let $a_n a_{n-1} \dots a_1 a_0$ be an $(n+1)$ - digit number. Then $4|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $4|(a_0 + 2a_1)$.

Proof:

Note that $a_n a_{n-1} \dots a_1 a_0 = a_0 * 10^0 + a_1 * 10^1 + \dots + a_{n-1} * 10^{n-1} + a_n * 10^n$.

Since $10^0 \equiv 1 \; mod\; 4$, $a_0 * 10^0 \equiv a_0 * 1 \equiv a_0 \; mod\; 4$.

Since $10^1 \equiv 2 \; mod\; 4$, $a_1 * 10^1 \equiv a_1 * 2 \equiv 2a_1 \; mod\; 4$.

Since $10^2 \equiv 0 \; mod\; 4$, $a_2 * 10^2 \equiv a_2 * 0 \equiv 0 \; mod\; 4$.

For $n > 2$, since $10^n \equiv 0 \; mod\; 4$, $a_n * 10^n \equiv a_n * 0 \equiv 0 \; mod\; 4$.

Then, $a_n a_{n-1} \dots a_1 a_0 = a_0 * 10^0 + a_1 * 10^1 + \dots + a_{n-1} * 10^{n-1} + a_n * 10^n$

$$\equiv a_0 + 2a_1 + 0 + \dots + 0 + 0$$

$$\equiv a_0 + 2a_1 \; mod\; 4.$$

Thus, $4|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $4|(a_0 + 2a_1)$.

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According to the above test, whether a number is divisible by 4 only depends on the last two digits of that number.

As an example, if we want to know whether 372 is divisible by 4, we check the last two digits which is 72. Using our notation, $a_0 = 2$ and $a_1 = 7$. Since $a_0 + 2a_1 = 2 + 2(7) = 16$ is divisible by 4, we know 372 is divisible by 4.

Below is a table showing the combination of the last two digits that when put together is divisible by 4. As a result, any number with such last two digits is divisible by 4.

a_0 $(mod\ 4)$	a_1 $(mod\ 2)$
0	0
2	1

These combinations work because when you multiply a_1 by 2, then add a_0, the sum of the remainders will be divisible by 4.

More specifically if $a_0 = 0, 4, \text{or } 8$, then a_1 would have to be an even number. Some of these numbers include 28, 40, 64, and 180, which are all divisible by 4.

In addition, if $a_0 = 2$ or 6, then a_1 would have to be an odd number. Some of these numbers include 16, 72, 136, and 152, which are all divisible by 4.

2.3: Divisibility Test for 8

Similar to the two previous tests, this test has been slightly touched on from [3], but below is a rigorous proof behind it.

Let $a_n a_{n-1} \dots a_1 a_0$ be an $(n + 1)$ - digit number. Then $8|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $8|(a_0 + 2a_1 + 4a_2)$.

Proof:

Since $10^0 \equiv 1 \, mod \, 8$, $a_0 * 10^0 \equiv a_0 * 1 \equiv a_0 \, mod \, 8$.

Since $10^1 \equiv 2 \, mod \, 8$, $a_1 * 10^1 \equiv a_1 * 2 \equiv 2a_1 \, mod \, 8$.

Since $10^2 \equiv 4 \, mod \, 8$, $a_2 * 10^2 \equiv a_2 * 4 \equiv 4a_2 \, mod \, 8$.

Since $10^3 \equiv 0 \, mod \, 8$, $a_3 * 10^3 \equiv a_3 * 0 \equiv 0 \, mod \, 8$.

For $n > 3$, since $10^n \equiv 0 \, mod \, 8$, $a_n * 10^n \equiv a_n * 0 \equiv 0 \, mod \, 8$.

Then, $a_n a_{n-1} \dots a_1 a_0 = a_0 * 10^0 + a_1 * 10^1 + \dots + a_{n-1} * 10^{n-1} + a_n * 10^n$

$$\equiv a_0 + 2a_1 + 4a_2 + 0 + \dots + 0 + 0$$

$$\equiv a_0 + 2a_1 + 4a_2 \, mod \, 8.$$

Thus, $8|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $8|(a_0 + 2a_1 + 4a_2)$.

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According to the test above, whether a number is divisible by 8 only depends on the last three digits of that number.

As an example, if we want to know whether 128 is divisible by 8, we check all 3 digits, where $a_0 = 8$, $a_1 = 2$, and $a_2 = 1$. Since $a_0 + 2a_1 + 4a_2 = 8 + 2(2) + 4(1) = 16$ is divisible by 8, we know 128 is divisible 8.

Below is a table showing the combination of the last three digits that when put together is divisible by 8. As a result, any number with such last three digits is divisible by 8.

$a_0 \ (mod\ 8)$	$a_1 \ (mod\ 4)$	$a_2 \ (mod\ 2)$
0	0	0
0	2	1
2	1	1
2	3	0
4	0	1
4	2	0
6	1	0
6	3	1

These combinations work because when you multiply a_1 by 2 and a_2 by 4, then add them together along with a_0, the sum of the remainders will be divisible by 8.

We will not go through all the cases, but as an example, let's look at the fourth row of the above table. In this case, $a_0 = 2$, $a_1 = 3\ or\ 7$, and a_2 is even. Some of these numbers include 232, 672, and 57,032, which are all divisible by 8.

2.4: Divisibility Test for 3 and 9

Note that 10 is not divisible by 3 or 9 because they would both lead to a remainder of 1, so we would have to perform a different test than the previous numbers.

Let $a_n a_{n-1} \dots a_1 a_0$ be an $(n+1)$ - digit number. Then $3|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $3|(a_0 + a_1 + \dots + a_{n-1} + a_n)$. Also, $9|(a_n a_{n-1} \dots a_1 a_0)$ if and only if

$9|(a_0+a_1 + \dots + a_{n-1}+a_n)$.

Proof:

Since $10^0 \equiv 1 \bmod 3$, $a_0 * 10^0 \equiv a_0 * 1 \equiv a_0 \bmod 3$.

Similarly, since $10^0 \equiv 1 \bmod 9$, $a_0 * 10^0 \equiv a_0 * 1 \equiv a_0 \bmod 9$.

Since $10^1 \equiv 1 \bmod 3$, $a_1 * 10^1 \equiv a_1 * 1 \equiv a_1 \bmod 3$.

Similarly, since $10^1 \equiv 1 \bmod 9$, $a_1 * 10^1 \equiv a_1 * 1 \equiv a_1 \bmod 9$.

Since $10^2 \equiv 1 \bmod 3$, $a_2 * 10^2 \equiv a_2 * 1 \equiv a_2 \bmod 3$.

Similarly, since $10^2 \equiv 1 \bmod 9$, $a_2 * 10^2 \equiv a_2 * 1 \equiv a_2 \bmod 9$.

Then, $a_n a_{n-1} \dots a_1 a_0 = a_0 * 10^0 + a_1 * 10^1 + \cdots + a_{n-1} * 10^{n-1} + a_n * 10^n$

$$\equiv a_0 + a_1 + \cdots + a_{n-1} + a_n \bmod 3.$$

Similarly, $a_n a_{n-1} \dots a_1 a_0 = a_0 * 10^0 + a_1 * 10^1 + \cdots + a_{n-1} * 10^{n-1} + a_n * 10^n$

$$\equiv a_0 + a_1 + \cdots + a_{n-1} + a_n \bmod 9.$$

Thus, $3|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $3|(a_0 + a_1 + \dots + a_{n-1} + a_n)$, and $9|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $9|(a_0 + a_1 + \dots + a_{n-1} + a_n)$.

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First, we look at 456, where $a_0 = 6$, $a_1 = 5$, and $a_2 = 4$. Since

$a_0 + a_1 + a_2 = 6 + 5 + 4 = 15$, and 15 is divisible by 3, we know 456 is divisible by 3. However, since 15 is not divisible by 9, 456 is not divisible by 9.

Next, we look at 729, where $a_0 = 9$, $a_1 = 2$, and $a_2 = 7$. Since

$a_0 + a_1 + a_2 = 9 + 2 + 7 = 18$, and 18 is divisible by 3, we know 729 is divisible by 3. Similarly, since 18 is divisible by 9, 729 is divisible by 9.

2.5: Divisibility Test for 11

When 10 is divided by 11, it gives a remainder of 10, or equivalently, -1. As a result, when 10^n is divided by 11, the remainder is $(-1)^n$. This fact allows us to establish the first divisibility test for 11.

First Divisibility Test for 11:

Let $a_n a_{n-1} \dots a_1 a_0$ be an $(n+1)$ - digit number. Then, $11|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $11|(a_0 - a_1 + a_2 - a_3 + \cdots)$.

Proof:

Since $10^0 \equiv 1 \; mod \; 11$, $a_0 * 10^0 \equiv a_0 * 1 \equiv a_0 \; mod \; 11$.

Since $10^1 \equiv -1 \; mod \; 11$, $a_1 * 10^1 \equiv a_1 * (-1) \equiv -a_1 \; mod \; 11$.

Since $10^2 \equiv 1 \; mod \; 11$, $a_2 * 10^2 \equiv a_2 * 1 \equiv a_2 \; mod \; 11$.

Since $10^3 \equiv -1 \; mod \; 11$, $a_3 * 10^3 \equiv a_3 * (-1) \equiv -a_3 \; mod \; 11$.

Then, $a_n a_{n-1} \dots a_1 a_0 = a_0 * 10^0 + a_1 * 10^1 + \cdots + a_{n-1} * 10^{n-1} + a_n * 10^n$

$$= a_0 - a_1 + a_2 - a_3 + \cdots \; mod \; 11.$$

Thus, $11|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $11|(a_0 - a_1 + a_2 - a_3 + \cdots)$.

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For example, with 198, let $a_0 = 8$, $a_1 = 9$, and $a_2 = 1$. Since $a_0 - a_1 + a_2 = 8 - 9 + 1 = 0$ is divisible by 11, 198 is also divisible by 11.

Similarly, when 100 is divided by 11, the remainder is 1. So, when 100^n is divisible by 11, then remainder is 1^n, which is always 1. Hence, we have our second divisibility test for 11.

Second Divisibility Test for 11:

$11|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $11|(a_1 a_0 + a_3 a_2 + a_5 a_4 + \cdots)$.

Proof:

Since $10^0 \equiv 1 \; mod \; 11$, $a_1 a_0 * 10^0 \equiv a_1 a_0 * 1 \equiv a_1 a_0 \; mod \; 11$.

Since $10^2 \equiv 1 \; mod \; 11$, $a_3 a_2 * 10^2 \equiv a_3 a_2 * 1 \equiv a_3 a_2 \; mod \; 11$.

Since $10^4 \equiv 1 \; mod \; 11$, $a_5 a_4 * 10^4 \equiv a_5 a_4 * 1 \equiv a_5 a_4 \; mod \; 11$.

Then, $a_n a_{n-1} \dots a_1 a_0 = a_1 a_0 * 10^0 + a_3 a_2 * 10^2 + a_5 a_4 * 10^4 + \cdots$

$$\equiv a_1 a_0 + a_3 a_2 + a_5 a_4 + \cdots \; mod \; 11.$$

Thus, $11|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $11|(a_1 a_0 + a_3 a_2 + a_5 a_4 + \cdots)$.

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Since the digits are grouped two at a time, each two-digit number is multiplied by 10, or –1, to an even power. Every time –1 is raised to an even power, it turns positive. This is why we can determine if a number is divisible by 11 by just adding the digits together two at a time.

If we have 1,210, then we can just add $a_1 a_0 = 10$ and $a_3 a_2 = 12$. Since

$a_1 a_0 + a_3 a_2 = 10 + 12 = 22$ is divisible by 11, 1,210 is divisible by 11.

Similarly, when 1,000 is divisible by 11, the remainder is –1. So, when 1000^n is divisible by 11, then the remainder is $(-1)^n$. Hence, we have our third divisibility test for 11.

Third Divisibility Test for 11:

$11|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $11|(a_2 a_1 a_0 - a_5 a_4 a_3 + a_8 a_7 a_6 - \cdots)$.

Proof:

Since $10^0 \equiv 1 \ mod\ 11$, $a_2 a_1 a_0 * 10^0 \equiv a_2 a_1 a_0 * 1 \equiv a_2 a_1 a_0 \ mod\ 11$.

Since $10^3 \equiv -1 \ mod\ 11$, $a_5 a_4 a_3 * 10^3 \equiv a_5 a_4 a_3 * (-1) \equiv -a_5 a_4 a_3 \ mod\ 11$.

Since $10^6 \equiv 1 \ mod\ 11$, $a_8 a_7 a_6 * 10^6 \equiv a_8 a_7 a_6 * 1 \equiv a_8 a_7 a_6 \ mod\ 11$.

Then, $a_n a_{n-1} \dots a_1 a_0 = a_2 a_1 a_0 * 10^0 + a_5 a_4 a_3 * 10^3 + a_8 a_7 a_6 * 10^6 + \cdots$

$$\equiv a_2 a_1 a_0 - a_5 a_4 a_3 + a_8 a_7 a_6 - \dots \ mod\ 11.$$

Thus, $11|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $11|(a_2 a_1 a_0 - a_5 a_4 a_3 + a_8 a_7 a_6 - \cdots)$.

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Like the first test, the digits alternate signs when they are grouped three at a time.

If we have 24,100,087, we can write it as 024,100,087, where $a_2 a_1 a_0 = 087$,

$a_5 a_4 a_3 = 100$, and $a_8 a_7 a_6 = 024$. Since $a_2 a_1 a_0 - a_5 a_4 a_3 + a_8 a_7 a_6 = 087 - 100 + 024 = 11$ is divisible by 11, 24,100,087 is also divisible by 11.

It is interesting to note that the third divisibility test for 11 also applies to 7 and 13. In other words, $7|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $7|(a_2 a_1 a_0 - a_5 a_4 a_3 + a_8 a_7 a_6 - \cdots)$, and $13|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $13|(a_2 a_1 a_0 - a_5 a_4 a_3 + a_8 a_7 a_6 - \cdots)$. This is due to the fact that when multiplying 7, 11, and 13 together, the product is 1,001, which means when 1,000 is divided by 7, 11, or 13, the remainder is –1. The proof of the above test is similar for 7 and 13 in terms of numbers with more digits. This is explained in [2] and [3] when the articles also showed the 7 * 11 * 13 = 1,001 method along with examples.

3. Main Result

To set the stage for the main result, we will first dive deeper into the divisibility test for 7. Note that the divisibility test at the end of the last chapter is beneficial for numbers with more than three digits. For a three-digit number, we have the following test, which is found on [2] and [3]. While they give different strategies on how to find the divisibility test for 7, the proof below gives a more detailed explanation about why it works. As a reminder, *abc* and *ab* below represent a 3-digit and 2-digit numbers respectively, not a product.

3.1: Divisibility Test for 7: a Warmup

Let abc be a 3-digit number. Then, $7|\ abc$ if and only if $7|\ (ab - 2c)$.

Proof:

Assume that $7|\ abc$. Since $abc = 10ab + c$, $7|\ (10ab + c)$.

That is, $10ab + c = 7k$ where k is an integer.

We can solve c to get $c = 7k - 10ab$. Then,

$$\begin{aligned} ab - 2c &= ab - 2\ (7k - 10ab) \\ &= ab - 14k + 20ab \\ &= -14k + 21ab \\ &= 7\ (-2k + 3ab) \end{aligned}$$

Since $-2k + 3ab$ is an integer, $7|\ (ab - 2c)$.

Now assume $7|\ (ab - 2c)$.

Then, $ab - 2c = 7k$ where k as an integer.

We can solve ab to get $ab = 7k + 2c$. Then,

$$\begin{aligned} 10ab + c &= 10\ (7k + 2c) + c \\ &= 70k + 20c + c \\ &= 70k + 21c \\ &= 7\ (10k + 3c). \end{aligned}$$

Since $10k + 3c$ is an integer, $7|\ (10ab + c)$. That is, $7|\ abc$.

Thus, $7|\ abc$ if and only if $7|\ (ab - 2c)$.

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As an example, let's take 343 and have $c = 3$ and $ab = 34$. Since $ab - 2c = 34 - 2(3) = 28$ is divisible by 7, 343 is also divisible by 7.

3.2: Main Result and Two Proofs

Let m be a number where the greatest common factor $(m, 10) = 1$, and let $a_n a_{n-1} \dots a_1 a_0$ be an $(n + 1)$ - digit number. If z is the multiplicative inverse of 10 mod m, then $m|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $m|(a_n a_{n-1} \dots a_1 + z a_0)$.

We will provide two different proofs. The first proof is similar to the above proof of the divisibility test for 7, and the second proof uses modular algebra.

Proof 1:

Assume that z is the multiplicative inverse of 10 mod m and $m|(a_n a_{n-1} \dots a_1 a_0)$.

Since $a_n a_{n-1} \dots a_1 a_0 = 10 a_n a_{n-1} \dots a_1 + a_0, m|(10 a_n a_{n-1} \dots a_1 + a_0)$.

Then, $10 a_n a_{n-1} \dots a_1 + a_0 = mk$, where k is an integer.

We can solve a_0 to get $a_0 = mk - 10 a_n a_{n-1} \dots a_1$. Then,

$$\begin{aligned} a_n a_{n-1} \dots a_1 + z a_0 &= a_n a_{n-1} \dots a_1 + z\,(mk - 10 a_n a_{n-1} \dots a_1) \\ &= a_n a_{n-1} \dots a_1 + zmk - 10 z a_n a_{n-1} \dots a_1 \\ &= zmk + (1 - 10z) a_n a_{n-1} \dots a_1. \end{aligned}$$

Since z is the multiplicative inverse of 10 mod m, then $10z \equiv 1 \; mod \; m$.

That is, $1 - 10z = mr$, with m as an integer.

$$\begin{aligned} \text{By substitution, } a_n a_{n-1} \dots a_1 + z a_0 &= zmk + mr a_n a_{n-1} \dots a_1 \\ &= m\,(zk + r a_n a_{n-1} \dots a_1). \end{aligned}$$

Since $zk + r a_n a_{n-1} \dots a_1$ is an integer, $m|(a_n a_{n-1} \dots a_1 + z a_0)$.

Now assume $m|(a_n a_{n-1} \dots a_1 + z a_0)$.

Then $a_n a_{n-1} \dots a_1 + z a_0 = mk$, with k as an integer.

We can solve $a_n a_{n-1} \dots a_1$ to get $a_n a_{n-1} \dots a_1 = mk - z a_0$. Then,

$$\begin{aligned} 10 a_n a_{n-1} \dots a_1 + a_0 &= 10\,(mk - z a_0) + a_0 \\ &= 10mk - 10 z a_0 + a_0 \\ &= 10mk + (1 - 10z) a_0. \end{aligned}$$

Since z is the multiplicative inverse of 10 mod m, then $10z \equiv 1 \text{ mod } m$.

That is, $1 - 10z = mr$, with m as an integer.

By substitution, $a_n a_{n-1} \dots a_1 = 10mk + mra_0$

$$= m\,(10k + ra_0).$$

Since $10k + ra_0$ is an integer, $m|(a_n a_{n-1} \dots a_1 a_0)$.

Therefore, $m|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $m|(a_n a_{n-1} \dots a_1 + za_0)$.

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Proof 2:

Assume that z is the multiplicative inverse of 10 mod m. To simplify our notation, let $x = a_0$ and $y = a_n a_{n-1} \dots a_1$. We must show that $10y + x \equiv 0 \bmod m$ if and only if $y + zx \equiv 0 \bmod m$.

Assume $10y + x \equiv 0 \bmod m$.

Multiplying both sides by z, we get

$z\,(10y + x) \equiv z\,(0) \bmod m$

$10zy + zx \equiv 0 \bmod m.$

Since $10z \equiv 1 \bmod m$,

$y + zx \equiv 0 \bmod m.$

Now, assume $y + zx \equiv 0 \bmod m$.

Multiplying both sides by 10, we get

$10\,(y + zx) \equiv 10\,(0) \bmod m$

$10y + 10zx \equiv 0 \bmod m.$

Since $10z \equiv 1 \bmod m$,

$10y + x \equiv 0 \bmod m.$

Since we proved $10y + x \equiv 0 \bmod m$ if and only if $y + zx \equiv 0 \bmod m$, substituting

$x = a_0$ and $y = a_n a_{n-1} \dots a_1$ back, we have $m|(10a_n a_{n-1} \dots a_1 + a_0)$ if and only if $m|(a_n a_{n-1} \dots a_1 + za_0)$. That is, $m|(a_n a_{n-1} \dots a_1 a_0)$ if and only if $m|(a_n a_{n-1} \dots a_1 + za_0)$.

■

3.3: Applications of the Main Result

In this section, we discuss applications of our main result to several specific numbers represented by m.

Application 1:

Let $m = 13$, we know the greatest common factor $(13, 10) = 1$. The multiplicative inverse of 10 mod 13 is $z = 4$ since $10 * 4 = 40 \equiv 1 \, mod \, 13$. Using our main result, if $a_n a_{n-1} \ldots a_1 a_0$ is a $(n + 1)$ - digit number, then $13|(a_n a_{n-1} \ldots a_1 a_0)$ if and only if $13|(a_n a_{n-1} \ldots a_1 + 4a_0)$.

We will use a three-digit number to demonstrate this result. For example, let's take 169 and have $a_0 = 9$ and $a_2 a_1 = 16$. Then, $a_2 a_1 + 4a_0 = 16 + 4(9) = 16 + 36 = 52$. Since 52 is divisible by 13, 169 is divisible by 13.

As another example, let's take 531 and have $a_0 = 1$ and $a_2 a_1 = 53$. Then,

$a_2 a_1 + 4a_0 = 53 + 4(1) = 53 + 4 = 57$. Since 57 is not divisible by 13, 531 is not divisible by 13.

This is reflected on [1].

Application 2:

Let $m = 29$, we know the greatest common factor $(29, 10) = 1$. The multiplicative inverse of 10 mod 29 is $z = 3$ since $10 * 3 = 30 \equiv 1 \, mod \, 29$. Using our main result, if $a_n a_{n-1} \ldots a_1 a_0$ is a $(n + 1)$ - digit number, then $29|(a_n a_{n-1} \ldots a_1 a_0)$ if and only if $29|(a_n a_{n-1} \ldots a_1 + 3a_0)$.

Just like the first application, we will use a three-digit number to demonstrate this statement. For example, let's take 438 and have $a_0 = 8$ and $a_2 a_1 = 43$. Then,

$a_2 a_1 + 3a_0 = 43 + 3(8) = 43 + 24 = 67$. Since 67 is not divisible by 29, 438 is not divisible by 29.

As another example, let's take 754 and have $a_0 = 4$ and $a_2 a_1 = 75$. Then,

$a_2 a_1 + 3a_0 = 75 + 3(4) = 75 + 12 = 87$. Since 87 is divisible by 29, 754 is divisible by 29.

Application 3:

Let $m = 83$, we know the greatest common factor $(83, 10) = 1$. The multiplicative inverse of 10 mod 83 is $z = 25$ since $10 * 25 = 250 \equiv 1 \, mod \, 83$. Using our main result, if $a_n a_{n-1} \ldots a_1 a_0$ is an $(n + 1)$ - digit number, then $83|(a_n a_{n-1} \ldots a_1 a_0)$ if and only if $83|(a_n a_{n-1} \ldots a_1 + 25a_0)$.

We will use a four-digit number to demonstrate the above statement. For example, let's take 1,256 and have $a_0 = 6$ and $a_3a_2a_1 = 125$. Then,

$a_3a_2a_1 + 25a_0 = 125 + 25(6) = 125 + 150 = 275$. Now we can use the same test to determine if 275 is divisible by 83. Since $a_0 = 5$ and $a_2a_1 = 27$, then,

$a_2a_1 + 25a_0 = 27 + 25(5) = 27 + 125 = 152$. Since 152 is not divisible by 83, 275 is not divisible by 83. Therefore 1,256 is not divisible by 83.

As another example, let's take 3,901 and have $a_0 = 1$ and $a_3a_2a_1 = 390$. Then,

$a_3a_2a_1 + 25a_0 = 390 + 25(1) = 390 + 25 = 415$. Now, let $a_0 = 5$ and

$a_2a_1 = 41$. Then, $a_2a_1 + 25a_0 = 41 + 25(5) = 41 + 125 = 166$. Since 166 is divisible by 83, 415 is divisible by 83. Therefore 3,901 is also divisible by 83.

4. Divisibility Tests in the Octal System

4.1: Background of the Octal System

We unconsciously use the decimal system or base 10 whenever we are counting, but how would counting work if we had a different base? The octal system, or base 8, uses digits 0 to 7 to count. To avoid confusion, we use $(a_n a_{n-1} \ldots a_1 a_0)_8$ to represent an $(n + 1)$ - digit number in the octal system, while we continue to use $a_n a_{n-1} \ldots a_1 a_0$ to represent an $(n + 1)$ - digit number in the decimal system. Note that in the octal system, the next number after 7_8 is 10_8, and the next number after 17_8 is 20_8.

Since 1, 10, 100, and 1,000 are powers of 10 in the decimal system, that means 1_8, 10_8, 100_8, and $1{,}000_8$ in the octal system are powers of 8 in the decimal system. Specifically,

1_8 in the octal system is 8^0, or 1, in the decimal system.

10_8 in the octal system is 8^1, or 8, in the decimal system.

100_8 in the octal system is 8^2, or 64, in the decimal system.

$1{,}000_8$ in the octal system is 8^3, or 512, in the decimal system.

To convert a number from the octal system to the decimal system, we take a number $(a_n a_{n-1} \ldots a_1 a_0)_8$ from the octal system and express it as the sum

$a_0 \cdot 8^0 + a_1 \cdot 8^1 + a_2 \cdot 8^2 + \cdots + a_n \cdot 8^n$. Let's take a number, $1{,}237_8$, in the octal system. Since $a_0 = 7$, we would multiply 7 by 1. Since $a_1 = 3$, we multiply 3 by 8. Since $a_2 = 2$, we multiply 2 by 8^2 or 64. Finally since $a_3 = 1$, we multiply 1 by 8^3 or 512. This makes

$1{,}237_8 = 7 + 3 * 8 + 2 * 64 + 512 = 671$ in the decimal system.

To convert a number from the decimal system to the octal system is a little more complicated. We would take a number from the decimal system and divide it by 8. The remainder would be the last digit of the octal system, and if the quotient is at least 8, we would repeat the same process until the quotient becomes 0 with the remainder becoming the first digit. For instance, if we take 671 in the decimal system and divide it by 8, the remainder is 7 which is what the last digit will be in the octal system. Meanwhile, we will continue to divide by 8 using the quotient 83. The remainder turns out to be 3 which is the next digit in the octal system while the new quotient is 10. If we divide it by 8 one more time, the remainder and next digit is 2 while the quotient is 1. Since 1 is less than 8, 1 is the final remainder and first digit of the octal number. This means $671 = 1{,}237_8$.

4.2: Divisibility Tests for 2 and 4

The divisibility tests for 2 and 4 in the octal system are similar to the divisibility tests for 2 and 5 in the decimal system because 2 and 4 are factors of 8.

Let $(a_n a_{n-1} \dots a_1 a_0)_8$ be an $(n + 1)$ - digit number in the octal system. Then $2|(a_n a_{n-1} \dots a_1 a_0)_8$ if and only if $2|a_0$. Similarly, $4|(a_n a_{n-1} \dots a_1 a_0)_8$ if and only if $4|a_0$.

Proof:

Since $8^0 \equiv 1 \bmod 2$, $a_0 * 8^0 \equiv a_0 * 1 \equiv a_0 \bmod 2$.

Similarly, since $8^0 \equiv 1 \bmod 4$, $a_0 * 8^0 \equiv a_0 * 1 \equiv a_0 \bmod 4$.

Since $8^1 \equiv 0 \bmod 2$, $a_1 * 8^1 \equiv a_1 * 0 \equiv 0 \bmod 2$.

Similarly, since $8^1 \equiv 0 \bmod 4$, $a_1 * 8^1 \equiv a_1 * 0 \equiv 0 \bmod 4$.

For $n > 1$, since $8^n \equiv 0 \bmod 2$, $a_n * 8^n \equiv a_n * 0 \equiv 0 \bmod 2$.

Similarly for $n > 1$, since $8^n \equiv 0 \bmod 4$, $a_n * 8^n \equiv a_n * 0 \equiv 0 \bmod 4$.

Then,
$$\begin{aligned}(a_n a_{n-1} \dots a_1 a_0)_8 &= a_0 * 8^0 + a_1 * 8^1 + \dots + a_{n-1} * 8^{n-1} + a_n * 8^n \\ &\equiv a_0 + 0 + 0 + \dots + 0 + 0 \\ &\equiv a_0 \bmod 2.\end{aligned}$$

Similarly,
$$\begin{aligned}(a_n a_{n-1} \dots a_1 a_0)_8 &= a_0 * 8^0 + a_1 * 8^1 + \dots + a_{n-1} * 8^{n-1} + a_n * 8^n \\ &\equiv a_0 + 0 + 0 + \dots + 0 + 0 \\ &\equiv a_0 \bmod 4.\end{aligned}$$

Thus, $2|(a_n a_{n-1} \dots a_1 a_0)_8$ if and only if $2|a_0$, and $4|(a_n a_{n-1} \dots a_1 a_0)_8$ if and only if $4|a_0$.

■

According to the test above, one can determine whether a number in the octal system is divisible by 2 or 4 by just looking at the last digit of that number.

For 2, the last digit would need to be 0, 2, 4, or 6.

For 4, the last digit must be 0 or 4.

As an example, if we want to know whether 36_8 is divisible by 2, we check the last digit which is 6. Since 6 is divisible by 2, we know 36_8 is divisible by 2.

As another example, if we want to know whether 44_8 is divisible by 4, we check the last digit which is 4. Since 4 is divisible by 4, 44_8 is divisible by 4.

4.3: Divisibility Test for 7

The divisibility for 7 in the octal system is similar to the divisibility for 9 in the decimal system because when 8 is divided by 7, it leaves a remainder of 1.

Let $(a_n a_{n-1} \dots a_1 a_0)_8$ be an $(n+1)$ - digit number in the octal system. Then $7|(a_n a_{n-1} \dots a_1 a_0)_8$ if and only if $7|(a_0 + a_1 + \dots + a_{n-1} + a_n)$.

Proof:

Since $8^0 \equiv 1 \; mod \; 7$, $a_0 * 8^0 \equiv a_0 * 1 \equiv a_0 \; mod \; 7$.

Since $8^1 \equiv 1 \; mod \; 7$, $a_1 * 8^1 \equiv a_1 * 1 \equiv a_1 \; mod \; 7$.

Since $8^2 \equiv 1 \; mod \; 7$, $a_2 * 8^2 \equiv a_2 * 1 \equiv a_2 \; mod \; 7$.

Then, $(a_n a_{n-1} \dots a_1 a_0)_8 = a_0 * 8^0 + a_1 * 8^1 + \cdots + a_{n-1} * 8^{n-1} + a_n * 8^n$

$$\equiv a_0 + a_1 + \cdots + a_{n-1} + a_n \; mod \; 7.$$

Thus, $7|(a_n a_{n-1} \dots a_1 a_0)_8$ if and only if $7|(a_0 + a_1 + \dots + a_{n-1} + a_n)$.

■

This means all we have to do is add all the digits together to find out if an octal number is divisible by 7. For example, let's look at 124_8. If we add the digits together, 1 + 2 + 4 = 7. Since 7 is divisible of 7, 124_8 is divisible by 7.

To verify this in the decimal system, note that $124_8 = 4 + 2 * 8 + 1 * 8^2 = 84$, which is indeed divisible by 7.

4.4: Divisibility Test for 9

The divisibility test for 9 in the octal system is similar to the divisibility test for 11 in the decimal system because when 8 is divided by 9, it leaves a remainder of -1. Similarly, we also have three different tests for 9 in the octal system, but we will only look at one here.

Let $(a_n a_{n-1} \dots a_1 a_0)_8$ be an $(n+1)$ - digit number in the octal system. Then, $9|(a_n a_{n-1} \dots a_1 a_0)_8$ if and only if $9|(a_0 - a_1 + a_2 - a_3 + \cdots)$.

Proof:

Since $8^0 \equiv 1 \; mod \; 9$, $a_0 * 8^0 \equiv a_0 * 1 \equiv a_0 \; mod \; 9$.

Since $8^1 \equiv -1 \; mod \; 9$, $a_1 * 8^1 \equiv a_1 * (-1) \equiv -a_1 \; mod \; 9$.

Since $8^2 \equiv 1 \; mod \; 9$, $a_2 * 8^2 \equiv a_2 * 1 \equiv a_2 \; mod \; 9$.

Since $8^3 \equiv -1 \; mod \; 9$, $a_3 * 8^3 \equiv a_3 * (-1) \equiv -a_3 \; mod \; 9$.

$$\begin{aligned} \text{Then, } (a_n a_{n-1} \dots a_1 a_0)_8 &= a_0 * 8^0 + a_1 * 8^1 + \cdots + a_{n-1} * 8^{n-1} + a_n * 8^n \\ &\equiv a_0 * (-1)^0 + a_1 * (-1)^1 + \cdots + a_n * (-1)^n mod \; 9 \\ &= a_0 - a_1 + a_2 - a_3 + \cdots \; mod \; 9. \end{aligned}$$

Thus, $9|(a_n a_{n-1} \dots a_1 a_0)_8$ if and only if $9|(a_0 - a_1 + a_2 - a_3 + \cdots)$.

■

For the first example, we will look at 526_8 with $a_0 = 6$, $a_1 = 2$, and $a_2 = 5$. Since $a_0 - a_1 + a_2 = 6 - 2 + 5 = 9$ is divisible by 9, 526_8 is also divisible by 9.

Next, we will look at $3{,}150_8$ with $a_0 = 0$, $a_1 = 5$, $a_2 = 1$, and $a_3 = 3$. Since $a_0 - a_1 + a_2 - a_3 = 0 - 5 + 1 - 3 = -7$ is not divisible by 9, $3{,}150_8$ is not divisible by 9.

4.5: Main Result - the Octal Version

Let m be a number where the greatest common factor of (m, 8) = 1, and let $(a_n a_{n-1} \dots a_0)_8$ be an $(n + 1)$ - digit number in the octal system. If z is the multiplicative inverse of 8 mod m, then $m|(a_n a_{n-1} \dots a_0)_8$ if and only if $m|((a_n a_{n-1} \dots a_1)_8 + za_0)$. The proof for this is similar to the one for the decimal system.

Proof 1:

Assume that z is the multiplicative inverse of 8 mod m and $m|(a_n a_{n-1} \dots a_1 a_0)_8$.

Since $(a_n a_{n-1} \dots a_1 a_0)_8 = 8(a_n a_{n-1} \dots a_1)_8 + a_0$, $m|(8(a_n a_{n-1} \dots a_1)_8 + a_0)$.

Then, $8(a_n a_{n-1} \dots a_1)_8 + a_0 = mk$, where k is an integer.

We can solve a_0 to get $a_0 = mk - 8(a_n a_{n-1} \dots a_1)_8$. Then,

$$(a_n a_{n-1} \dots a_1)_8 + za_0 = (a_n a_{n-1} \dots a_1)_8 + z(mk - 8(a_n a_{n-1} \dots a_1)_8)$$

$$= (a_n a_{n-1} \dots a_1)_8 + zmk - 8z(a_n a_{n-1} \dots a_1)_8$$

$$= zmk + (1 - 8z)(a_n a_{n-1} \dots a_1)_8.$$

Since z is the multiplicative inverse of 8 mod m, then $8z \equiv 1 \bmod m$.

That is, $1 - 8z = mr$, with m as an integer.

By substitution, $(a_n a_{n-1} \dots a_1)_8 + za_0 = zmk + mr(a_n a_{n-1} \dots a_1)_8$

$$= m(zk + r(a_n a_{n-1} \dots a_1)_8).$$

Since $zk + r(a_n a_{n-1} \dots a_1)_8$ is an integer, $m|((a_n a_{n-1} \dots a_1)_8 + za_0)$.

Now assume $m|((a_n a_{n-1} \dots a_1)_8 + za_0)$.

Then $(a_n a_{n-1} \dots a_1)_8 + za_0 = mk$, with k as an integer.

We can solve $(a_n a_{n-1} \dots a_1)_8$ to get $(a_n a_{n-1} \dots a_1)_8 = mk - za_0$. Then,

$$(a_n a_{n-1} \dots a_0)_8 = 8(a_n a_{n-1} \dots a_1)_8 + a_0$$

$$= 8(mk - za_0) + a_0$$

$$= 8mk - 8za_0 + a_0$$

$$= 8mk + (1 - 8z)a_0.$$

Since z is the multiplicative inverse of 8 mod m, then $8z \equiv 1 \bmod m$.

That is, $1 - 8z = mr$, with m as an integer.

By substitution, $(a_n a_{n-1} \ldots a_1 a_0)_8 = 8mk + mra_0$

$$= m\,(8k + ra_0).$$

Since $8k + ra_0$ is an integer, $m|(a_n a_{n-1} \ldots a_1 a_0)_8$.

Therefore $m|(a_n a_{n-1} \ldots a_0)_8$ if and only if $m|((a_n a_{n-1} \ldots a_1)_8 + za_0)$.

■

Proof 2:

Assume that z is the multiplicative inverse of 8 mod m. To simplify our notation, let $x = a_0$ and $y = (a_n a_{n-1} \ldots a_1)_8$. We must show that $8y + x \equiv 0 \bmod m$ if and only if $y + zx \equiv 0 \bmod m$.

Assume $8y + x \equiv 0 \bmod m$.

Multiplying both sides by z, we get

$z\,(8y + x) \equiv z\,(0) \bmod m$

$8zy + zx \equiv 0 \bmod m$.

Since $8z \equiv 1 \bmod m$,

$y + zx \equiv 0 \bmod m$.

Now, assume $y + zx \equiv 0 \bmod m$.

Multiplying both sides by 8, we get

$8\,(y + zx) \equiv 8\,(0) \bmod m$

$8y + 8zx \equiv 0 \bmod m$.

Since $8z \equiv 1 \bmod m$,

$8y + x \equiv 0 \bmod m$.

Since we proved $8y + x \equiv 0 \; mod \; m$ if and only if $y + zx \equiv 0 \; mod \; m$, substituting

$x = a_0$ and $y = (a_n a_{n-1} \ldots a_1)_8$ back, we have $m|(8(a_n a_{n-1} \ldots a_1)_8 + a_0)$ if and only if $m|((a_n a_{n-1} \ldots a_1)_8 + za_0)$. That is, $m|(a_n a_{n-1} \ldots a_1 a_0)_8$ if and only if $m|((a_n a_{n-1} \ldots a_1)_8 + za_0)$.

■

4.6: Application: Divisibility Test for 13

In the decimal system, we provided three examples of applications, but we will only present one application for the octal system.

Let $m = 13$, we know the greatest common factor $(13, 8) = 1$. The multiplicative inverse of

$8 \; mod \; 13$ is $z = 5$ since $8 * 5 = 40 \equiv 1 \; mod \; 13$. Using our main result, if $(a_n a_{n-1} \ldots a_1 a_0)_8$ is a $(n + 1)$ - digit number, then $13|(a_n a_{n-1} \ldots a_1 a_0)_8$ if and only if

$13|((a_n a_{n-1} \ldots a_1)_8 + z a_0)$.

Let's take 111_8 with $a_0 = 1$ and $(a_2 a_1)_8 = 11_8$. Then, $(a_2 a_1)_8 + 5a_0 = 11_8 + 5 * 1 = 16_8$, which is equal to 14 in the decimal system. Since 14 is not divisible by 13, 111_8 is not divisible by 13.

Now, let's look at 202_8 with $a_0 = 2$ and $(a_2 a_1)_8 = 20_8$. Then, $(a_2 a_1)_8 + 5a_0 = 20_8 + 5 * 2$. It is important to note that $5 * 2 = 10$ in the decimal system, which is equal to 12_8 in the octal system. Thus, $(a_2 a_1)_8 + 5a_0 = 20_8 + 12_8 = 32_8$, which is equal to $2 + 3 * 8 = 26$ in the decimal system. Since 26 is divisible by 13, 202_8 is divisible by 13.

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